THE FINITE INTERVALS OF THE MUCHNIK LATTICE

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Abstract

We characterize the finite intervals of the Muchnik lattice by proving that they are a certain proper subclass of the finite distributive lattices.

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1 Introduction

The Medvedev lattice and the Muchnik lattice are structures from computability theory that were originally defined for their connections with constructive logic, but that are of independent interest as well. Both can be seen as generalizations of the Turing degrees, and for example when Muchnik presented his solution to Post's problem he phrased it as a result of the Medvedev lattice. In Terwijn [15] the structure of the Medvedev lattice $\mathfrak{M}$ was investigated, and it was proven there that the finite intervals of $\mathfrak{M}$ are precisely the finite Boolean algebras, and that the infinite intervals of $\mathfrak{M}$ all have cardinality $2^{2^{\aleph_0}}$ (cf. Theorem 1.5). It was noted there that this strong dichotomy does not hold for the Muchnik lattice $\mathfrak{M}_w$, and that there are many more possibilities for intervals in $\mathfrak{M}_w$, both for the finite and for the infinite ones. In this paper we characterize the finite intervals of $\mathfrak{M}_w$ by proving that they are a certain subclass of the finite distributive lattices that can be described using elementary lattice theory. In the rest of this section we will repeat the necessary definitions and some further preliminaries.

The Medvedev lattice, introduced by Medvedev [5], is a particular way of specifying Kolmogorov idea of a calculus of problems. Let ω denote the natural numbers and let ω^ω be the set of all functions from ω to ω (Baire space). A *mass problem* is a subset of ω^ω. Every mass problem is associated with the "problem" of producing an element of it. A mass problem $\mathcal{A}$ *Medvedev reduces* to mass problem $\mathcal{B}$, denoted $\mathcal{A} \leqslant_M \mathcal{B}$, if there is a partial computable functional $\Psi : \omega^\omega \to \omega^\omega$ defined on all of

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$\mathcal{B}$ such that $\Psi(\mathcal{B}) \subseteq \mathcal{A}$. That is, Ψ is a uniformly effective method for transforming solutions to $\mathcal{B}$ into solutions to $\mathcal{A}$. The relation $\leqslant_M$ induces an equivalence relation on mass problems: $\mathcal{A} \equiv_M \mathcal{B}$ if $\mathcal{A} \leqslant_M \mathcal{B}$ and $\mathcal{B} \leqslant_M \mathcal{A}$. The equivalence class of $\mathcal{A}$ is denoted by $\deg_M(\mathcal{A})$ and is called the *Medvedev degree* of $\mathcal{A}$. We denote Medvedev degrees by boldface symbols. There is a smallest Medvedev degree, denoted by $\mathbf{0}$, namely the degree of any mass problem containing a computable function, and there is a largest degree $\mathbf{1}$, the degree of the empty mass problem, of which it is absolutely impossible to produce an element. A meet operator $\times$ and a join operator $+$ are defined on mass problems as follows: For functions f and g, as usual define the function $f \oplus g$ by $f \oplus g(2x) = f(x)$ and $f \oplus g(2x+1) = g(x)$. Let $n\widehat{\ }\mathcal{A} = \{n\widehat{\ }f : f \in \mathcal{A}\}$, where $\widehat{\ }$ denotes concatenation. Define

$$\mathcal{A} + \mathcal{B} = \big\{ f \oplus g : f \in \mathcal{A} \wedge g \in B \big\}$$

and

$$\mathcal{A} \times \mathcal{B} = 0\widehat{\ }\mathcal{A} \cup 1\widehat{\ }\mathcal{B}.$$

The structure $\mathfrak{M}$ of all Medvedev degrees, ordered by $\leqslant_M$ and together with $+$ and $\times$ is a distributive lattice (Medvedev [5]).

The Muchnik lattice, introduced by Muchnik [7], is a nonuniform variant of the Medvedev lattice. It is the structure $\mathfrak{M}_w$ resulting from the reduction relation on mass problems defined by

$$\mathcal{A} \leqslant_w \mathcal{B} \equiv (\forall f \in \mathcal{B})(\exists g \in \mathcal{A})[g \leqslant_T f].$$

(The "w" stands for "weak".) That is, every solution to the mass problem $\mathcal{B}$ can compute a solution to the mass problem $\mathcal{A}$, but maybe not in a uniform way. It is easy to check that $\mathfrak{M}_w$ is a distributive lattice in the same way that $\mathfrak{M}$ is, with the same lattice operations and $\mathbf{0}$ and $\mathbf{1}$. Notice that $\mathcal{A} \times \mathcal{B}$ in $\mathfrak{M}_w$ simplifies to $\mathcal{A} \cup \mathcal{B}$.

An M-degree is a *Muchnik degree* if it contains a mass problem that is upwards closed under Turing reducibility $\leqslant_T$. The Muchnik degrees of $\mathfrak{M}$ form a substructure that is isomorphic to $\mathfrak{M}_w$. For any mass problem $\mathcal{A}$, let $C(\mathcal{A})$ denote the upward closure of $\mathcal{A}$ under $\leqslant_T$. We have the following embeddings:

$$\begin{array}{ccccc}
\mathcal{D}_T & \hookrightarrow & \mathfrak{M}_w & \hookrightarrow & \mathfrak{M} \\
\text{Turing degrees} & & \text{Muchnik degrees} & & \text{Medvedev degrees} \\
\\
\deg_T(f) & \longmapsto & \deg_w(\{f\}) & & \\
\\
& & \deg_w(\mathcal{A}) & \longmapsto & \deg_M(C(\mathcal{A}))
\end{array}$$

More discussion about the elementary properties of $\mathfrak{M}$ and $\mathfrak{M}_w$ can be found in in Rogers' textbook [10] and the survey paper by Sorbi [13]. Previous results about

embeddings of lattices and algebras into $\mathfrak{M}$ and $\mathfrak{M}_w$ can be found in Sorbi [11, 12]. Binns and Simpson [1] contains results about lattice embeddings into the lattice of Π^0_1-classes under $\leqslant_M$ and $\leqslant_w$.

Our notation is mostly standard and follows Odifreddi [8]. Φ_e is the e-th partial computable functional. For countable sets $I \subseteq \omega$ and mass problems $\mathcal{A}_i$, $i \in I$, we have the meet operator

$$\textstyle\prod_{i\in I}\mathcal{A}_i = \big\{i\widehat{\ }f : i \in I \wedge f \in \mathcal{A}_i\big\}.$$

Note that for finite I this is M-equivalent to an iteration of the meet operator $\times$. If $a \leqslant b$ in some partial order, we use the interval notation $[a, b] = \{x : a \leqslant x \leqslant b\}$. Similarly (a, b) denotes an interval without endpoints, and $(a]$ denotes the set $\{x : x \leqslant a\}$. We say that b *covers* a if $b > a$ and there is no x with $a < x < b$.

In the final section of [15] some consequences of the results of that paper for the Muchnik lattice $\mathfrak{M}_w$ were listed. Some of these consequences were:

- In contrast to $\mathfrak{M}$, the lattice $\mathfrak{M}_w$ contains nonempty linear intervals.
- Every finite Boolean algebra is isomorphic to an interval of $\mathfrak{M}_w$.
- Every Muchnik degree is as large as set-theoretically possible: For every mass problem $\mathcal{F} \subseteq 2^\omega$ we have $|\deg_{M_w}(\mathcal{F})| = 2^{2^{\aleph_0}}$.
- Whereas in $\mathfrak{M}$ only countable Boolean algebras can be embedded, the dual of $\mathcal{P}(2^\omega)$ is embeddable into $\mathfrak{M}_w$ as a Boolean algebra.

A Medvedev degree is a *degree of solvability* if it contains a singleton mass problem. A mass problem is called *nonsolvable* if its M-degree is not a degree of solvability. For every degree of solvability $\mathbf{S}$ there is a unique minimal M-degree $> \mathbf{S}$ that is denoted by $\mathbf{S}'$ (cf. Medvedev [5]). If $\mathbf{S} = \deg_M(\{f\})$ then $\mathbf{S}'$ is the degree of the mass problem

$$\{f\}' = \big\{n\widehat{\ }g : f <_T g \wedge \Phi_n(g) = f\big\}.$$

Note that $\{\emptyset\}'$ is M-equivalent to the set of all noncomputable functions. We will also denote this set by $0'$. Note further that $\{f\}' \equiv_w \{g \in \omega^\omega : f <_T g\}$ so that in $\mathfrak{M}_w$ we can use this simplified version of $\{f\}'$. Dyment [2] proved that the degrees of solvability are precisely characterized by the existence of such an $\mathbf{S}'$. Namely, the degrees of solvability are first-order definable (both in $\mathfrak{M}$ and in $\mathfrak{M}_w$) by the formula

$$\phi(x) = \exists y\big(x < y \wedge \forall z(x < z \rightarrow y \leqslant z)\big).$$

Thus the Turing degrees form a first-order definable substructure of both $\mathfrak{M}$ and $\mathfrak{M}_w$. This has many immediate corollaries, for example that the first-order theories of the structures $(\mathfrak{M}, \leqslant_M)$ and $(\mathfrak{M}_w, \leqslant_w)$ are undecidable.

Theorem 1.1. (Dyment [2], cf. [15, Theorem 2.5]) *For Medvedev degrees* $\mathbf{A}$ *and* $\mathbf{B}$ *with* $\mathbf{A} <_M \mathbf{B}$ *it holds that* $(\mathbf{A}, \mathbf{B}) = \emptyset$ *if and only if there is a degree of solvability* $\mathbf{S}$ *such that* $\mathbf{A} \equiv_M \mathbf{B} \times \mathbf{S}$*,* $\mathbf{B} \not\leqslant_M \mathbf{S}$*, and* $\mathbf{B} \leqslant_M \mathbf{S}'$*.*

Theorem 1.1 also holds for $\mathfrak{M}_w$, with a much easier proof. We will include a proof here, as a warm-up for Section 3.

Lemma 1.2. *Suppose that* $\mathcal{A}$ *and* $\mathcal{B}$ *satisfy*

$$\forall \mathcal{C} \subseteq \mathcal{A} \textit{ finite } \; (\mathcal{B} \times \mathcal{C} \not\leqslant_w \mathcal{A}). \tag{1}$$

Then there exists $\mathcal{C} \geqslant_w \mathcal{A}$ *such that* $\mathcal{C} \not\geqslant_w \mathcal{B}$ *and* $\mathcal{B} \times \mathcal{C} \not\leqslant_w \mathcal{A}$*. If moreover* $\mathcal{A} \leqslant_w \mathcal{B}$ *then the interval* $(\mathcal{A}, \mathcal{B})$ *is infinite.*

Proof. Since from (1) it follows that $\mathcal{B} \not\leqslant_w \mathcal{A}$, there is $f \in \mathcal{A}$ such that $\{f\} \not\geqslant_w \mathcal{B}$. Again by (1) we have that $\mathcal{B} \times \{f\} \not\leqslant_w \mathcal{A}$, so we can take $\mathcal{C} = \{f\}$.

If in addition $\mathcal{A} \leqslant_w \mathcal{B}$ then we have $\mathcal{A} <_w \mathcal{B} \times \{f\} <_w \mathcal{B}$. Since $\mathcal{A}$ and $\mathcal{B} \times \{f\}$ also satisfy (1) we can by iteration of the first part of the lemma obtain an infinite downward chain in $(\mathcal{A}, \mathcal{B})$. □

Theorem 1.3. (Dyments Theorem for $\mathfrak{M}_w$) *For Muchnik degrees* $\mathbf{A}$ *and* $\mathbf{B}$ *with* $\mathbf{A} <_w \mathbf{B}$ *it holds that* $(\mathbf{A}, \mathbf{B}) = \emptyset$ *if and only if there is a degree of solvability* $\mathbf{S}$ *such that* $\mathbf{A} \equiv_w \mathbf{B} \times \mathbf{S}$*,* $\mathbf{B} \not\leqslant_w \mathbf{S}$*, and* $\mathbf{B} \leqslant_w \mathbf{S}'$*.*

Proof. (If) Suppose that $\mathbf{S} = \deg_w(\{f\})$ is as in the theorem and suppose that $\mathcal{A} \in \mathbf{A}$, $\mathcal{B} \in \mathbf{B}$, and $\mathcal{B} \times \{f\} \leqslant_M \mathcal{C} \leqslant_M \mathcal{B}$. If $\mathcal{C}$ does not contain any element of Turing degree $\deg_T(f)$ then it follows that $\mathcal{C} \geqslant_w \mathcal{B} \times \{f\}'$, because the elements of $\mathcal{C}$ that get sent to the $\{f\}$-side are all strictly above f, hence included in $\{f\}'$. So in this case $\mathcal{C} \geqslant_w \mathcal{B}$ by $\{f\}' \geqslant_w \mathcal{B}$.

Otherwise $\mathcal{C}$ contains an element of Turing degree $\deg_T(f)$, and consequently $\mathcal{C} \leqslant_w \{f\}$. Hence $\mathcal{C} \leqslant_w \mathcal{B} \times \{f\} \equiv_w \mathcal{A}$.

(Only if) Suppose that $(\mathcal{A}, \mathcal{B}) = \emptyset$. Then by Lemma 1.2, $\mathcal{A}$ and $\mathcal{B}$ do not satisfy condition (1), hence there is a finite set $\mathcal{C} \subseteq \mathcal{A}$ such that $\mathcal{B} \times \mathcal{C} \leqslant_w \mathcal{A}$. There is also an $f \in \mathcal{C}$ such that $\{f\} \not\geqslant_w \mathcal{B}$, for otherwise we would have $\mathcal{A} \geqslant_w \mathcal{B}$. Because the interval is empty and $\mathcal{A} \leqslant_w \mathcal{B} \times \{f\} <_w \mathcal{B}$ we must have $\mathcal{A} \equiv_w \mathcal{B} \times \{f\}$ since there is no other possibility for $\mathcal{B} \times \{f\}$. We also have $\mathcal{B} \times \{f\}' \not\leqslant_w \mathcal{A}$ because both $\{f\} \not\geqslant_w \mathcal{B}$ and $\{f\} \not\geqslant_w \{f\}'$. Hence $\mathcal{B} \times \{f\}' \equiv_w \mathcal{B}$, again by emptiness of the interval, and in particular $\{f\}' \geqslant_w \mathcal{B}$. So we can take $\mathbf{S}$ to be $\deg_w(\{f\})$. □

Let f and g be T-incomparable. Then the interval $\big[\{f, g\}, \{f\}' \times \{g\}'\big]$ contains exactly two intermediate elements, cf. Figure 1. This can be generalized to obtain finite intervals of size 2^n for any n as follows:

$$
\begin{tikzcd}[column sep=small]
 & \{f\}' \times \{g\}' \arrow[dl, dash] \arrow[dr, dash] & \\
\{f\} \times \{g\}' \arrow[dr, dash] & & \{g\} \times \{f\}' \arrow[dl, dash] \\
 & \{f, g\} &
\end{tikzcd}
$$

Figure 1

Theorem 1.4. *Let $\mathcal{B}$ be any mass problem. Let $n \geqslant 1$ and let $f_1, \ldots, f_n \in \omega^\omega$ be T-incomparable such that $\{f_i\} \not\geqslant_w \mathcal{B}$ for every i. Then the interval*

$$\left[\mathcal{B} \times \{f_1, \ldots, f_n\}, \mathcal{B} \times \{f_1\}' \times \ldots \times \{f_n\}'\right]$$

in $\mathfrak{M}_w$ is isomorphic to the finite Boolean algebra $\mathbf{2}^n$.

Proof. This was proved in [15] for $\mathfrak{M}$. It holds for $\mathfrak{M}_w$ with the same proof. $\square$

Platek [9] proved that $\mathfrak{M}$ has the (for a collection of sets of reals maximal possible) cardinality $2^{2^{\aleph_0}}$ by showing that $\mathfrak{M}$ has antichains of that cardinality. (He mentions that the result was noted independently by Elisabeth Jockusch and John Stillwell.) In fact, in $\mathfrak{M}$ such large antichains occur in every infinite interval:

Theorem 1.5. (Terwijn [15]) *Let $[\mathbf{A}, \mathbf{B}]$ be an interval in $\mathfrak{M}$ with $\mathbf{A} <_M \mathbf{B}$. Then either $[\mathbf{A}, \mathbf{B}]$ is isomorphic to the Boolean algebra $\mathbf{2}^n$ for some $n \geqslant 1$, or $[\mathbf{A}, \mathbf{B}]$ contains an antichain of size $2^{2^{\aleph_0}}$. In the latter case, assuming CH, it also contains a chain of size $2^{2^{\aleph_0}}$.*

In particular, $\mathfrak{M}$'s version of Theorem 1.4 is the *only* way to generate finite intervals of $\mathfrak{M}$. As we will see in what follows, the situation for $\mathfrak{M}_w$ is rather different.

2 More on chains and antichains

Although every countable linear order can be embedded into $\mathfrak{M}_w$ (because by Lachlan (cf. [8, p529]) this already holds for the Turing degrees), the following result shows that not every countable linear order is *isomorphic* to an interval in $\mathfrak{M}_w$. (From Theorem 3.12 it will follow that every finite linear order is isomorphic to an interval in $\mathfrak{M}_w$.)

Proposition 2.1. *Not every countable linear order is isomorphic to an interval in $\mathfrak{M}_w$.*

Proof. Consider the linear order $\omega + \omega^*$ (that is, a copy of ω followed by a reverse copy of ω). Suppose that $\mathcal{A}_n$ and $\mathcal{B}_n$, $n \in \omega$, are mass problems such that for all n and m

$$\mathcal{A}_n <_w \mathcal{A}_{n+1} <_w \mathcal{B}_{m+1} <_w \mathcal{B}_m.$$

Let $\mathcal{C} = \prod_{m\in\omega}\mathcal{B}_m \equiv_w \bigcup_{m\in\omega}\mathcal{B}_m$. Then for all n, $\mathcal{A}_n <_w \mathcal{C} <_w \mathcal{B}_n$, so the interval $[\mathcal{A}_0, \mathcal{B}_0]$ is not isomorphic to $\omega + \omega^*$. $\square$

Proposition 2.2. *$\mathfrak{M}_w$ contains linear intervals that are countably infinite.*

Proof. By Lachlan, cf. [8, p529], every countable distributive lattice with a least element is embeddable as an initial segment in the Turing degrees. Consider the linear order $1 + \omega^*$ (a least element plus a reverse copy of ω) and embed this in the T-degrees: Let $\mathcal{F} = \{f_n : n \in \omega\}$ be such that $f_{n+1} <_T f_n$ and such that

$$h \leqslant_T f_0 \rightarrow h \text{ computable } \vee \exists n\, h \equiv_T f_n$$

for every h. Let $\mathcal{B} = \{h : h \not\leqslant_T f_0\}$ and $\mathcal{A} = \mathcal{B} \times \mathcal{F}$. (Note that $\mathcal{A}$ is in fact w-equivalent to $0'$.) Now if $\mathcal{C} \in [\mathcal{A}, \mathcal{B}]$ then $\mathcal{C}$ can be split in disjoint parts $\mathcal{C}_0$ and $\mathcal{C}_1$ such that $\mathcal{C}_0$ is maximal with the property $\mathcal{B} \leqslant_w \mathcal{C}_0$ and $\{f : f \leqslant_T f_0\} \leqslant_w \mathcal{C}_1$. Then $\mathcal{C}_1 \subseteq \mathcal{F}$ and $\mathcal{C} \equiv_w \mathcal{B} \times \mathcal{C}_1$. So it suffices to analyze all subclasses of $\mathcal{F}$: For every $I \subseteq \omega$ consider $\mathcal{C}_I = \mathcal{B} \times \{f_n : n \in I\}$. If I is infinite then $\mathcal{C}_I \leqslant_w \mathcal{F}$, hence $\mathcal{C}_I \equiv_w \mathcal{A}$. For I and J finite we have $\mathcal{C}_I \leqslant_w \mathcal{C}_J$ whenever $\min I \leqslant \min J$. So we see that the interval $(\mathcal{A}, \mathcal{B})$ contains only the countably many elements $\mathcal{B} \times \{f_n\}$, $n \in \omega$. $\square$

By Proposition 2.2 there are linear nonempty intervals in $\mathfrak{M}_w$. This contrasts the situation for $\mathfrak{M}$, where by Theorem 1.5 all the linear intervals are empty. So here we already see that Theorem 1.4 is not the only way anymore to generate finite intervals.

$\mathfrak{M}_w$ contains antichains of size $2^{2^{\aleph_0}}$, using the same argument that Platek used for $\mathfrak{M}$, (starting with an antichain of size $2^{\aleph_0}$ in the Turing degrees, form $2^{2^{\aleph_0}}$ incomparable combinations) but Proposition 2.2 shows that they do not occur in every infinite interval, as we had for $\mathfrak{M}$ (cf. Theorem 1.5). In fact there are intervals with maximal antichains of every possible size:

Theorem 2.3. *Each of the following possibilities is realized by some interval $[\mathbf{A}, \mathbf{B}]$ in $\mathfrak{M}_w$:*

1. $[\mathbf{A}, \mathbf{B}]$ *contains an antichain of size n, but not of size $n+1$,*
2. $[\mathbf{A}, \mathbf{B}]$ *contains an antichain of size $\aleph_0$, but no uncountable antichain,*
3. $[\mathbf{A}, \mathbf{B}]$ *contains an antichain of size $2^{\aleph_0}$, but not of size $2^{2^{\aleph_0}}$,*
4. (Platek [9]) $[\mathbf{A}, \mathbf{B}]$ *contains an antichain of size $2^{2^{\aleph_0}}$.*

Proof. *Ad 1.* This follows from Theorem 1.4.

Ad 2. Let $x_0 < x_1 < x_2 < \dots$ be an increasing chain of elements in some lattice and let $y_0 > y_1 > y_2 < \dots$ be a decreasing chain of elements in the same lattice such that $x_n \mid y_n$ for all n. Let L be the free distributive lattice on these sets of elements with an additional least element. Then L is a countable bottomed distributive lattice, so by Lachlan [8, p529], L is embeddable into the Turing degrees as an initial segment. Let $\{f_n : n \in \omega\}$ and $\{g_n : n \in \omega\}$ be representatives from the image $\mathrm{Im}(L)$ of L corresponding to the sequences x_n and y_n respectively, so that $f_i |_T g_j$ for all i and j and such that for all n, $f_n <_T f_{n+1}$ and $g_{n+1} <_T g_n$. Let

$$\begin{aligned} \mathcal{B} &= \{h : \forall n \;\; h \not\leqslant_T f_n, g_n\} \text{ and} \\ \mathcal{A} &= \mathcal{B} \times \{f_n, g_n : n \in \omega\}. \end{aligned}$$

Then every $\mathcal{C} \in [\mathcal{A}, \mathcal{B}]$ can be split as $\mathcal{C} \equiv_w \mathcal{C}_0 \times \mathcal{C}_1$, with $\mathcal{C}_0 \subseteq \mathcal{C}$ maximal with the property that $\mathcal{B} \leqslant_w \mathcal{C}_0$ and $\mathcal{C}_1 \subseteq \mathrm{Im}(L)$. Claim: the only elements of $\mathrm{Im}(L)$ that are not in $\mathcal{B}$ are the f_n, g_n, $n \in \omega$. To see the claim, note that the nonzero elements of $\mathrm{Im}(L)$ are free combinations of the f_n and g_n. Clearly $\mathcal{B}$ is closed under joins. By freeness of L it also easily follows that $\mathrm{Im}(L) - \{f_n, g_n : n \in \omega\}$ is closed under meets. Hence $\mathrm{Im}(L) \cap \mathcal{B}$ is closed under meets and joins, and from this it easily follows by induction on the complexity of the elements that every element in $\mathrm{Im}(L) - \{f_n, g_n : n \in \omega\}$ is in $\mathcal{B}$. This proves the claim. As a consequence, we have (by maximality of $\mathcal{C}_0$) that $\mathcal{C}_1 \subseteq \{f_n, g_n : n \in \omega\}$. Now $\deg_w(\mathcal{C})$ is determined by $\deg_w(\mathcal{C}_1)$: One easily checks that if $\mathcal{C}, \mathcal{D} \in [\mathcal{A}, \mathcal{B}]$ are split as above as $\mathcal{C} \equiv_w \mathcal{C}_0 \times \mathcal{C}_1$ and $\mathcal{D} \equiv_w \mathcal{D}_0 \times \mathcal{D}_1$ then $\mathcal{C}_1 \equiv_w \mathcal{D}_1$ implies that $\mathcal{C} \equiv_w \mathcal{D}$. In its turn, $\deg_w(\mathcal{C}_1)$ is determined by the minimal n (if any) such that $f_n \in \mathcal{C}_1$ and by whether $\mathcal{C}_1$ contains infinitely or finitely many g_m's, and in the latter case by the maximal m (if any) such that $g_m \in \mathcal{C}_1$. So we see that there are only countably many possibilities for the degree of $\mathcal{C}_1$, and hence for the degree of $\mathcal{C}$, and hence $[\mathcal{A}, \mathcal{B}]$ is countable.

Now consider the mass problems $\mathcal{C}_n = \mathcal{B} \times \{f_n, g_n\}$. Clearly $\mathcal{C}_n \,|_w\, \mathcal{C}_m$ if $n \neq m$. So $[\mathcal{A}, \mathcal{B}]$ is countable and contains an infinite antichain.

Ad 3. Let L be a countably infinite distributive lattice with a least element and an infinite antichain. By Lachlan [8, p529], L is embeddable into the Turing degrees as an initial segment. Let f_n, $n \in \omega$, be a set of representatives of all the degrees in the image of L. Consider the interval $[\mathcal{A}, \mathcal{B}]$, where $\mathcal{B} = \{h : \forall n \; h \not\leqslant_T f_n\}$ and $\mathcal{A} = \mathcal{B} \times \{f_n : n \in \omega\}$. Then $[\mathcal{A}, \mathcal{B}]$ contains an infinite antichain of elements of the form $\mathcal{B} \times \{f\}$ because L contains a corresponding infinite antichain. For $I \subseteq \omega$ let $\mathcal{C}_I = \mathcal{B} \times \big\{f_n : n \in I\big\}$. Then for incomparable sets I, $J \subseteq \omega$ it holds that $\mathcal{C}_I \,|_w\, \mathcal{C}_J$. So $[\mathcal{A}, \mathcal{B}]$ contains an antichain of size $2^{\aleph_0}$. Now if $\mathcal{C} \in [\mathcal{A}, \mathcal{B}]$ then $\mathcal{C} \equiv_w \mathcal{C}_0 \times \mathcal{C}_1$, with $\mathcal{C}_0 \subseteq \mathcal{C}$ maximal with the property that $\mathcal{B} \leqslant_w \mathcal{C}_0$ and $\mathcal{C}_1 \subseteq \mathcal{A}$. So the Muchnik degree $\deg_w(\mathcal{C})$ of every $\mathcal{C} \in [\mathcal{A}, \mathcal{B}]$ is determined by a countable set $\mathcal{C}_1$, hence there are at most $2^{\aleph_0}$ many elements in $[\mathcal{A}, \mathcal{B}]$.

Ad 4. We can apply Plateks argument to any interval that contains an antichain of size $2^{\aleph_0}$ of singletons: Suppose that the interval $[\mathcal{A}, \mathcal{B}]$ contains the elements

$\mathcal{B} \times \{f_\alpha\}$, $\alpha < 2^\omega$, such that the f_α form an antichain in the Turing degrees. For $I \subseteq 2^\omega$ let $\mathcal{C}_I = \mathcal{B} \times \big\{f_\alpha : \alpha \in I\big\}$. Clearly $\mathcal{C}_I \in [\mathcal{A}, \mathcal{B}]$. Now for incomparable sets I, $J \subseteq 2^\omega$ it holds that $\mathcal{C}_I \,|_w\, \mathcal{C}_J$, so it suffices to note that there is an antichain of size $2^{2^{\aleph_0}}$ in $\mathcal{P}(2^\omega)$. (For some general notes on chains and antichains we refer to [15].) $\square$

From the proof of Theorem 2.3 we can also read off some consequences for chains in $\mathfrak{M}_w$:

1. By Theorem 1.4 there are intervals containing chains of size n but not of size $n+1$,
2. By the proof of item 2., and also Proposition 2.2, there are countable intervals with an infinite chain,
3. The example of an interval given in the proof of item 3. contains also a chain of size $2^{\aleph_0}$, but not of size $2^{2^{\aleph_0}}$. This is because $\mathcal{P}(\omega)$ has a chain of size $2^{\aleph_0}$ so the same holds with ω replaced by $\{f_n : n \in \omega\}$. A chain in the interval of item 3. cannot be bigger since the interval itself was of size $2^{\aleph_0}$.
4. CH implies that $\mathfrak{M}_w$ has a chain of size $2^{2^{\aleph_0}}$, cf. [15]. The conditions for the existence of chains of size $2^{2^{\aleph_0}}$ in $\mathcal{P}(2^\omega)$, in $\mathfrak{M}$, and in $\mathfrak{M}_w$ are the same. The consistency of the existence of chains of this size also follows from the fact that the dual of $\mathcal{P}(2^\omega)$ is embeddable into $\mathfrak{M}_w$, cf. [15].

3 The finite intervals of $\mathfrak{M}_w$

Theorem 3.1. (Sorbi [11, 13]) *A countable distributive lattice with 0,1 is embeddable into $\mathfrak{M}$ (preserving 0 and 1) if and only if 0 is meet-irreducible and 1 is join-irreducible.*

Sorbi proved Theorem 3.1 by embedding the (unique) countable dense Boolean algebra into $\mathfrak{M}$. Since this algebra is embeddable into $\mathfrak{M}$ it also embeds into $\mathfrak{M}_w$. (This is because $\leqslant_M$ is stronger than $\leqslant_w$ and because for any mass problems $\mathcal{A}$ and $\mathcal{B}$ the sets $\mathcal{A}+\mathcal{B}$ and $\mathcal{A}\times\mathcal{B}$ are the same in $\mathfrak{M}$ and in $\mathfrak{M}_w$.) In particular every finite distributive lattice is embeddable into $\mathfrak{M}_w$. In the following we consider lattices that are *isomorphic* to an interval of $\mathfrak{M}_w$. In Theorem 1.5 we saw that for $\mathfrak{M}$ these were precisely the finite Boolean algebras. Of course no nondistributive lattice can be isomorphic to an interval in $\mathfrak{M}$ or $\mathfrak{M}_w$ since both structures are distributive (Medvedev [5]). In this section we characterize the finite intervals of $\mathfrak{M}_w$ as a certain subclass of the finite distributive lattices (Theorem 3.12). We start with some illustrative examples.

Example 3.2. That the diamond lattice is isomorphic to an interval in $\mathfrak{M}_w$ was already shown in Theorem 1.4. For later purposes we show that this way of obtaining a diamond is essentially the only way. Suppose that $[\mathcal{A},\mathcal{D}]$ is an interval in $\mathfrak{M}_w$ containing precisely two intermediate elements $\mathcal{B}$ and $\mathcal{C}$, and that $\mathcal{B}$ and $\mathcal{C}$ are incomparable, cf. Figure 2. Then $\mathcal{A}$ and $\mathcal{D}$ do not satisfy property (1) of

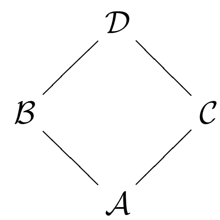

Figure 2

Lemma 1.2 because the interval is finite. So there is a finite set $\mathcal{X} \subseteq \mathcal{A}$ such that $\mathcal{D} \times \mathcal{X} \leqslant_w \mathcal{A}$, and hence $\mathcal{D} \times \mathcal{X} \equiv_w \mathcal{A}$. Without loss of generality the elements of $\mathcal{X}$ are pairwise T-incomparable and $\{f\} \not\geqslant_w \mathcal{D}$ for every $f \in \mathcal{X}$. Since for every $f \in \mathcal{X}$ we have $\mathcal{A} <_w \mathcal{D} \times \{f\} <_w \mathcal{D}$ we see that $\mathcal{X}$ can contain at most two elements. If $\mathcal{X}$ would contain only one element f then $[\mathcal{D} \times \{f\}, \mathcal{D} \times \{f\}']$ would be an initial segment of the interval $[\mathcal{A},\mathcal{D}]$, which is a contradiction. So $\mathcal{X}$ contains precisely two elements, f_0 and f_1 say. Then $[\mathcal{A},\mathcal{D}]$ contains the interval $[\mathcal{D}\times\{f_0,f_1\}, \mathcal{D}\times\{f_0\}'\times\{f_1\}']$, which by Theorem 1.4 is isomorphic to the diamond lattice. So we must have that $\mathcal{D} \equiv_w \mathcal{D} \times \{f_0\}' \times \{f_1\}'$. □

Example 3.3. We continue with Example 3.2. We show how to obtain the lattice depicted in Figure 3 as an interval. Let f_0 and f_1 be T-incomparable and let

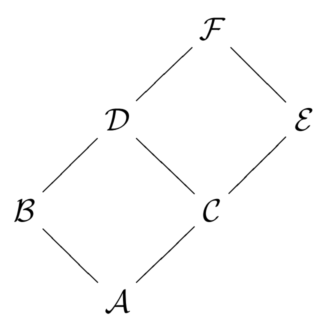

Figure 3

$g_1 |_T f_0$ be minimal over f_1, i.e. $g_1 >_T f_1$ and there is no function of T-degree strictly between f_1 and g_1. (In lattice theoretic terminology g_1 is said to *cover* f_1.) Let $\mathcal{X} = \{h \geqslant_T f_1 : h|_T g_1\}$. We then have $\mathcal{X} \times \{f_1\}' \equiv_w \mathcal{X} \times \{g_1\}$, as is easy to

check. Define

$$\begin{aligned}
\mathcal{A} &= \mathcal{X} \times \{f_0, f_1\} \\
\mathcal{B} &= \mathcal{X} \times \{f_0\}' \times \{f_1\} \\
\mathcal{C} &= \mathcal{X} \times \{f_0\} \times \{f_1\}' = \mathcal{X} \times \{f_0, g_1\} \\
\mathcal{D} &= \mathcal{X} \times \{f_0\}' \times \{f_1\}' = \mathcal{X} \times \{f_0\}' \times \{g_1\} \\
\mathcal{E} &= \mathcal{X} \times \{f_0\} \times \{g_1\}' \\
\mathcal{F} &= \mathcal{X} \times \{f_0\}' \times \{g_1\}'
\end{aligned}$$

Then by Example 3.2 the intervals $[\mathcal{A}, \mathcal{D}]$ and $[\mathcal{C}, \mathcal{F}]$ are both isomorphic to a diamond, hence the whole interval $[\mathcal{A}, \mathcal{F}]$ is isomorphic to the configuration of Figure 3. As in Example 3.2 one can also show that the above method is essentially the *only* way of obtaining this configuration as an interval of $\mathfrak{M}_w$. $\square$

Using similar methods as in the previous examples one can show that the lattices from Figures 4 and 5 can be obtained as intervals of $\mathfrak{M}_w$. For the first one uses

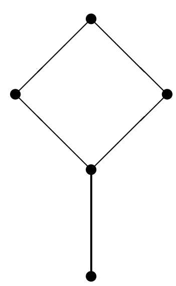

Figure 4

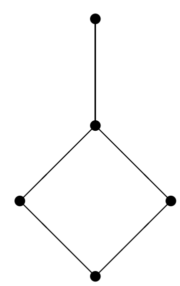

Figure 5

an f branching into two incomparable elements f_0 and f_1, and for the second one uses incomparable elements f_0 and f_1 and their join $f_0 \oplus f_1$.

Next we show that not every finite distributive lattice is isomorphic to an interval in $\mathfrak{M}_w$.

Proposition 3.4. *The double diamond lattice from Figure 6 is not isomorphic to an interval in* $\mathfrak{M}_w$.

Proof. Assume for a contradiction that the interval $[\mathcal{A}, \mathcal{G}]$ is isomorphic to the lattice of Figure 6. As in Example 3.2 we can argue that there is a finite set $\mathcal{X} \subseteq \mathcal{A}$ such that $\mathcal{A} \equiv_w \mathcal{X} \times \mathcal{G}$. Using the same reasoning as before we can argue that $\mathcal{X}$ contains precisely two T-incomparable elements f_0 and f_1 with $\{f_0\}, \{f_1\} \not\geq_w \mathcal{G}$. (If $\mathcal{X}$ contained at least three of such elements then by Theorem 1.4 the interval

$$
\begin{tikzcd}
& \mathcal{G} \arrow[dl, dash] \arrow[dr, dash] & \\
\mathcal{E} \arrow[dr, dash] & & \mathcal{F} \arrow[dl, dash] \\
& \mathcal{D} \arrow[dl, dash] \arrow[dr, dash] & \\
\mathcal{B} \arrow[dr, dash] & & \mathcal{C} \arrow[dl, dash] \\
& \mathcal{A} &
\end{tikzcd}
$$

Figure 6

$[\mathcal{A}, \mathcal{G}]$ would contain a copy of $\mathbf{2}^3$, but the interval contains only 7 elements so this is impossible.) Since by Example 3.2 there is only one way of obtaining a diamond, there are T-incomparable g_0 and g_1 with $\{g_0\}, \{g_1\} \not\geqslant_T \mathcal{G}$ such that

$$
\begin{aligned}
\mathcal{A} &\equiv_w \mathcal{G} \times \{f_0, f_1\} \\
\mathcal{B} &\equiv_w \mathcal{G} \times \{f_0\}' \times \{f_1\} \\
\mathcal{C} &\equiv_w \mathcal{G} \times \{f_0\} \times \{f_1\}' \\
\mathcal{D} &\equiv_w \mathcal{G} \times \{f_0\}' \times \{f_1\}' \equiv_w \mathcal{G} \times \{g_0, g_1\} \\
\mathcal{E} &\equiv_w \mathcal{G} \times \{g_0\}' \times \{g_1\} \\
\mathcal{F} &\equiv_w \mathcal{G} \times \{g_0\} \times \{g_1\}' \\
\mathcal{G} &\equiv_w \mathcal{G} \times \{g_0\}' \times \{g_1\}'
\end{aligned}
$$

From the two equations for $\mathcal{D}$ it follows that $\{g_0, g_1\} >_w \{f_0, f_1\}$. Now there are two cases:

- Both g_i's are T-above both f_j's. But then we have

  $$\mathcal{D} \equiv_w \mathcal{G} \times \{f_0\}' \times \{f_1\}' \leqslant_w \mathcal{G} \times \{f_0 \oplus f_1\} <_w \mathcal{G} \times \{g_0, g_1\} \equiv_w \mathcal{D},$$

  a contradiction. (The second to last inequality is strict since $f_0 \oplus f_1 <_T g_0, g_1$ because g_0 and g_1 are incomparable.)

- The g_i's are not both above f_0 and f_1. Hence either there is precisely one g_i above each f_j, or there are precisely two g_i's above one f_j. In both cases there is at least one g_i T-incomparable to an f_j, say that $f_0 |_T g_1$. Now consider the element $\mathcal{H} = \mathcal{G} \times \{f_0\} \times \{g_1\}'$. Clearly $\mathcal{H} \in [\mathcal{A}, \mathcal{G}]$. But $\mathcal{H}$ is w-incomparable to $\mathcal{D}$: $\mathcal{H} \not\leqslant_w \mathcal{D}$ because $\mathcal{H} \not\leqslant_w \{g_1\}$, and $\mathcal{D} \not\leqslant_w \mathcal{H}$ because $\mathcal{D} \not\leqslant_w \{f_0\}$. So again we have reached a contradiction, because $[\mathcal{A}, \mathcal{G}]$ does not contain an element incomparable to $\mathcal{D}$.

Since both cases are contradictory we conclude that it is impossible that $[\mathcal{A}, \mathcal{G}]$ is isomorphic to the double diamond. $\square$

We will see in Theorem 3.12 that the double diamond lattice of Figure 6 is the smallest possible counterexample.

Let us recall some elementary lattice theory from Grätzer [3]. Let L be a distributive lattice. $J(L)$ denotes the set of all *nonzero* join-irreducible elements of L. $J(L)$ is a poset under the ordering of L. For $a \in L$ define

$$r(a) = \{x \in J(L) : x \leqslant a\}.$$

For a poset P let $H(P)$ be the collection of downwards closed subsets of P, partially ordered by inclusion. Then $H(P)$ is a distributive lattice, and we have

Theorem 3.5. ([3, Theorem II.1.9]) *For any finite distributive lattice L the mapping $a \mapsto r(a)$ is an isomorphism between L and $H(J(L))$.*

Thus the mappings J and H are inverses of each other, and they relate the class of finite distributive lattices with the class of all finite posets.

Say that a lattice L contains another lattice L' as a *subinterval* if there is an interval $[a, b] \subseteq L$ such that $[a, b] \cong L'$. Note that this is not the same as saying that L' is a sublattice of L. For example, the free distributive lattice on three elements $F_D(3)$, depicted in Figure 7, contains the double diamond of Figure 6 as a sublattice, but not as a subinterval.

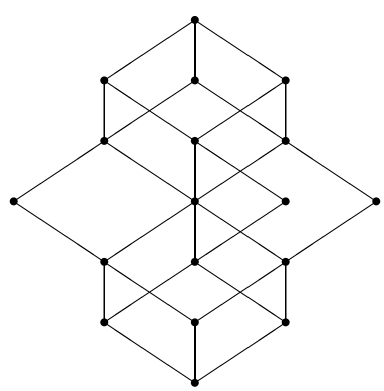

Figure 7

Definition 3.6. We call a finite distributive lattice *double diamond-like* if it has at least two elements, it has no largest and smallest nonzero join-irreducible element, and 0 is not the meet of two nonzero join-irreducible elements one of which is maximal.

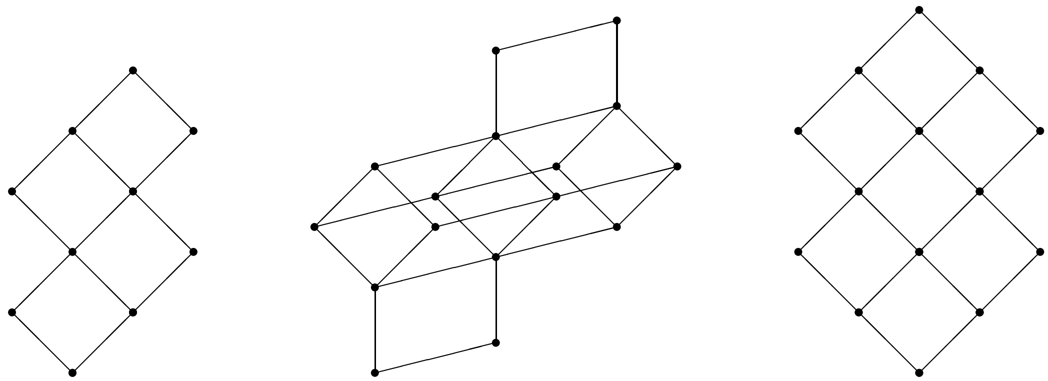

Figure 8: Some double diamond-like lattices.

The double diamond lattice from Figure 6 is the smallest example of a double diamond-like lattice. Figure 8 shows some other examples.

Theorem 3.7. *For a finite distributive lattice L, the poset $J(L)$ is an initial segment of an upper semilattice if and only if L has no double diamond-like subinterval.*

Proof. Suppose that $J(L)$ is an initial segment of an upper semilattice and let $[a, b]$ be an interval in L. We prove that $[a, b]$ is not double diamond-like. To this end, suppose that $a \neq b$ and that $J([a, b])$ has no largest and no smallest element. We have to prove that then the 0 of $[a, b]$, which is a, is a meet of two elements of $J([a, b])$, one of which is maximal. By the assumption $a \neq b$ we have $J([a, b]) \neq \emptyset$, and therefore there are at least two maximal nonzero join-irreducible elements y_0 and y_1 in $[a, b]$ and at least two minimal ones, x_0 and x_1 say. Suppose that both y's are above both x's, so that $J([a, b])$ contains the configuration of Figure 9. We

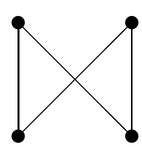

Figure 9

cannot immediately conclude from this that $J(L)$ contains the same configuration, for $J([a, b])$ and $J(L)$ can even be disjoint. Nevertheless, suppose that y_0 is join-reducible in L as $y_0 = z_0 + z_1$, with $z_0 | z_1$. By Lemma 3.8 we can choose z_0 and z_1 such that $z_0 \times x_0 \neq z_0 \times x_1$ and $z_0 \not\leq y_1$. Then in L the set $\{z_0 \times x_0, z_0 \times x_1, z_0, y_1\}$,

is partially ordered as in Figure 9. Continuing in this way we can reduce the configuration until the top element y_0 has become join-irreducible, and of course we can reduce y_1 in the same way. Then L contains the configuration of Figure 9 with both top elements in $J(L)$. But the bottom elements always bound nonzero join-reducible elements, so we see that L contains Figure 9 with all four elements in $J(L)$. But this contradicts that $J(L)$ is an initial of an upper semilattice and hence that the bottom two elements should have a least upper bound in $J(L)$. We conclude that in the original configuration $\{x_0, x_1, y_0, y_1\}$ in $J([a, b])$ it is not possible that both y's are above both x's, so there is at least one x that is not below a y, say $x_0 \,|\, y_1$. But then for the join-irreducible elements x_0 and y_1 in $[a, b]$ we have $a = x_0 \times y_1$: If $x_0 \times y_1 > a$ then x_0 would bound some join-irreducible element $> a$, contradicting that x_0 is minimal in $J([a, b])$. This shows that $[a, b]$ is not double diamond-like.

Conversely, if L is a lattice such that $J(L)$ is not an initial segment of any upper semilattice then it must contain two incomparable elements x_0 and x_1 without a least upper bound. If x_0 and x_1 would have no or only one upper bound in $J(L)$ then $J(L)$ could be consistently extended to an upper semilattice, so there must be at least *two* incomparable minimal upper bounds y_0 and y_1 for both x_0 and x_1. But then the poset $\{x_0, x_1, y_0, y_1\}$ is isomorphic to the configuration in Figure 9 (with possibly other elements in between the x's and y's). Denote this subposet of $J(L)$ by P. Now it is not hard to check that L has a double diamond-like subinterval. Namely by Theorem 3.5 we have $L \cong H(J(L))$. Consider the interval $[X, Y]$ in $H(J(L))$ defined by

$$\begin{aligned} X &= \big\{x \in J(L) : x < x_0 \vee x < x_1\big\}, \\ Y &= \big\{x \in J(L) : x \leqslant y_0 \vee x \leqslant y_1\big\}. \end{aligned}$$

Then clearly $[X, Y]$ is double diamond-like, because $X \neq Y$ and $[X, Y] = H(P)$ and so $J([X, Y]) = J(H(P)) \cong P$. □

Lemma 3.8. *In the proof of Theorem 3.7 above, if $y_0 = z_0 + z_1$ in L, $z_0|z_1$, then we can choose such z_0 and z_1 with $z_0 \times x_0 \neq z_0 \times x_1$ and $z_0 \nleqslant y_1$.*

Proof. Suppose that $y_0 = z_0 + z_1$ in L, with $z_0|z_1$. Note that z_0 and z_1 cannot be both in $[a, b]$. Suppose that

$$\forall v, w \in L\big(v|w \wedge y_0 = v + w \rightarrow v, w \notin [a, b]\big). \tag{2}$$

Consider z_0 and $a + z_1$. If $a + z_1 = y_0$ then this contradicts (2) (because both $a, z_1 < y_0$ they must be incomparable in this case). If $a + z_1 < y_0$ then by $(a + z_1) + z_0 = y_0$ we again contradict (2). Hence (2) is false, and if $y_0 = z_0 + z_1$ with $z_0|z_1$ in L we can always choose $z_0 \notin [a, b]$ and $z_1 \in [a, b]$. In this case

$z_0 + a = y_0$, for if $z_0 + a < y_0$ then by $(a + z_0) + z_1 = y_0$ we would have y_0 join-reducible in $[a, b]$, contradiction. Hence for every $c \in [a, y_0]$ it holds that $z_0 + c = y_0$, and in particular

$$z_0 + x_1 = z_0 + x_1 = y_0. \tag{3}$$

Now we also have $z_0 \nleqslant y_1$ because otherwise $y_0 = z_0 + x_0 \leqslant y_1$, contradiction.

Finally we prove that $z_0 \times x_0 \neq z_0 \times x_1$. Suppose that $z_0 \times x_0 = z_0 \times x_1$. Because by (3) it holds that $z_0 + x_1 = z_0 + x_1$ we have

$$\begin{aligned} x_1 &= (z_0 \times x_1) + x_1 \\ &= (z_0 \times x_0) + x_1 \\ &= (z_0 + x_1) \times (x_0 + x_1) \quad \text{(by distributivity)} \\ &= (z_0 + x_0) \times (x_0 + x_1) \\ &= x_0 + (z_0 \times x_1) \\ &\geqslant x_0 \end{aligned}$$

From this contradiction we conclude that $z_0 \times x_0 \neq z_0 \times x_1$. □

Example 3.9. Before giving the general result of how to obtain lattices as intervals of $\mathfrak{M}_w$ we give a specific example to illustrate the method. Figure 10 depicts the procedure to obtain a given lattice L as an interval of $\mathfrak{M}_w$. The top left side of the picture shows an example of a finite distributive lattice, with its nonzero join-irreducible elements circled. The partial order $J(L)$ is depicted on the top right. Now for the lattice L in this particular example we can map the poset $J(L)$ to an isomorphic configuration $I(J(L))$ in $\mathcal{D}_T$. (The picture remains the same, so we drew it only once.) This means that the only relations are the ones indicated in the picture, g_0 covers f_0 and f_1, and g_1 covers f_1. Next we can form the distributive lattice $H = H(I(J(L))$, which is isomorphic to L by Theorem 3.5. Finally we apply the mapping $F : H \to \mathfrak{M}_w$ defined as follows. First define

$$\mathcal{X} = \big\{h \geqslant_T f_0 : h|_T g_0\big\} \cup \big\{h \geqslant_T f_1 : h|_T g_0 \wedge h|_T g_1\big\}.$$

This has the effect that modulo $\mathcal{X}$ we have $\mathcal{X} \times \{f_0\}' \equiv_w \mathcal{X} \times \{g_0\}$ and $\mathcal{X} \times \{f_1\}' \equiv_w \mathcal{X} \times \{g_0, g_1\}$. For every $A \in H$ define

$$\widehat{A} = \big\{f \in I(J(L)) : f \text{ maximal in } A\big\}.$$

Finally define

$$F(A) = \mathcal{X} \times \prod\big\{\{f\}' : f \in \widehat{A}\big\} \times \big\{f \in I(J(L)) : f|_T \widehat{A}\big\}.$$

Here $f|_T \widehat{A}$ denotes that $f|_T g$ for every $g \in \widehat{A}$. We thus obtain the lattice $F(H)$

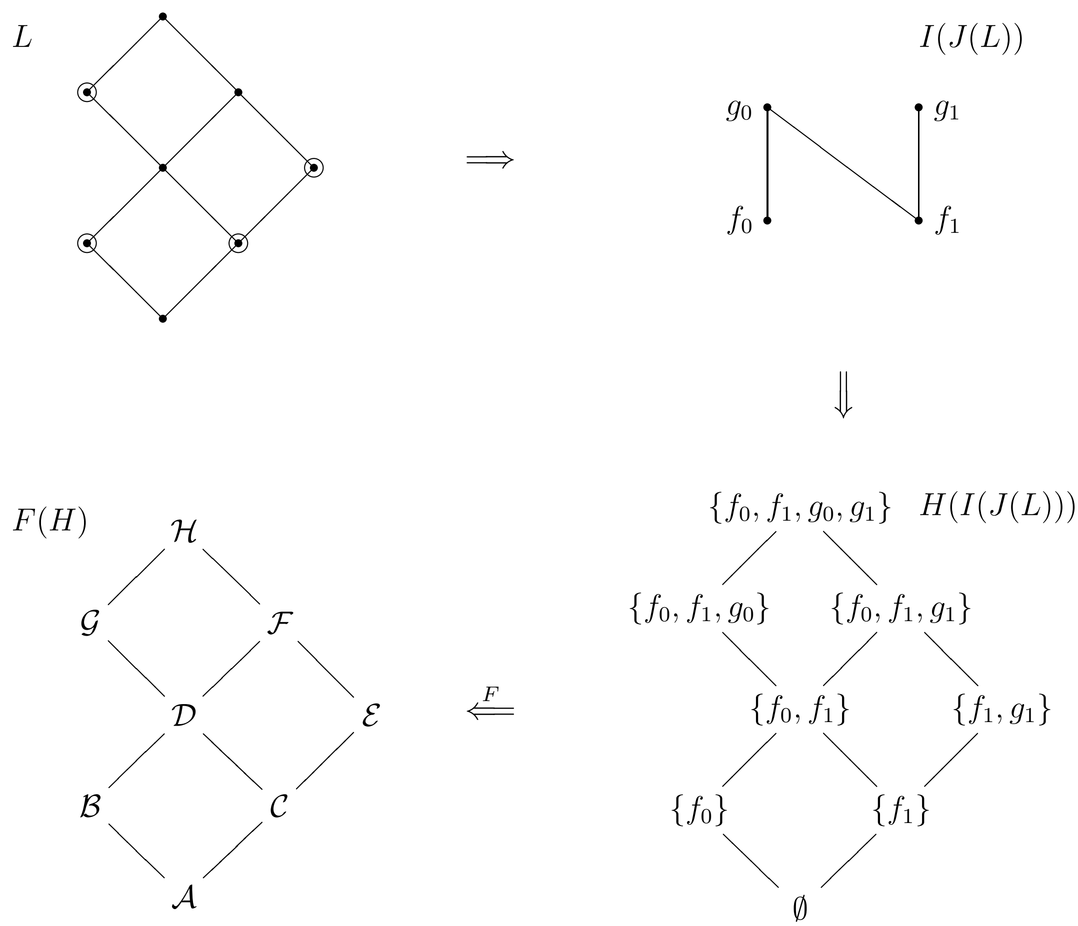

Figure 10: Procedure to obtain an interval in $\mathfrak{M}_w$ isomorphic to a given L.

on the bottom left of the picture, with

$$\begin{aligned}
\mathcal{A} &= \mathcal{X} \times \{f_0, f_1\} \\
\mathcal{B} &= \mathcal{X} \times \{f_0\}' \times \{f_1\} \\
\mathcal{C} &= \mathcal{X} \times \{f_0\} \times \{f_1\}' = \mathcal{X} \times \{f_0, g_1\} \\
\mathcal{D} &= \mathcal{X} \times \{f_0\}' \times \{f_1\}' = \mathcal{X} \times \{f_0\}' \times \{g_1\} = \mathcal{X} \times \{g_0, g_1\} \\
\mathcal{E} &= \mathcal{X} \times \{f_0\} \times \{g_1\}' \\
\mathcal{F} &= \mathcal{X} \times \{f_0\}' \times \{g_1\}' = \mathcal{X} \times \{g_0\} \times \{g_1\}' \\
\mathcal{G} &= \mathcal{X} \times \{g_0\}' \times \{g_1\} \\
\mathcal{H} &= \mathcal{X} \times \{g_0\}' \times \{g_1\}'
\end{aligned}$$

Using Examples 3.2 and 3.3 one can check that F is an isomorphism between H and $F(H)$, so that the interval $\big[\mathcal{A}, \mathcal{H}\big] = \big[F(\emptyset), F(I(J(L)))\big]$ is indeed isomorphic to L. $\square$

We are now ready to prove:

Theorem 3.10. *Suppose that L is a finite distributive lattice such that $J(L)$ is a an initial segment of a finite upper semilattice. Then L is isomorphic to an interval of $\mathfrak{M}_w$.*

Proof. We follow the procedure depicted in Figure 10. Let L be as in the hypothesis of the theorem. Since every finite upper semilattice with a least element is isomorphic to an initial segment of the Turing degrees $\mathcal{D}_T$ (cf. Lerman [4, p156]) we have a finite poset $I(J(L))$ in $\mathcal{D}_T$ that is isomorphic to $J(L)$. This means that if that if g covers f in $J(L)$ then the image of g is a minimal cover of f in $\mathcal{D}_T$. Furthermore, the minimal elements of $I(J(L))$ can be chosen to be of minimal T-degree (so that in particular they are all noncomputable). Next we form the distributive lattice $H = H(I(J(L)))$, which is isomorphic to L by Theorem 3.5. Finally we define the mapping $F : H \to \mathfrak{M}_w$ as follows. For a given $f \in I(J(L))$ let $g_0, \dots, g_m$ be all elements of $I(J(L))$ covering f. Define

$$\mathcal{X}_f = \big\{ h \in \omega^\omega : h >_T f \wedge h|_T g_0 \wedge \ldots \wedge h|_T g_m \big\},$$

$$\mathcal{X} = \bigcup_{f \in I(J(L))} \mathcal{X}_f.$$

Notice that if f is maximal in $I(J(L))$ then there are no elements of $I(J(L))$ covering f, hence $\mathcal{X}_f \equiv_w \{f\}'$. Next, for every $A \in H$ define

$$\widehat{A} = \big\{ f \in I(J(L)) : f \text{ maximal in } A \big\},$$

$$F(A) = \mathcal{X} \times \prod \big\{ \{f\}' : f \in \widehat{A} \big\} \times \big\{ f \in I(J(L)) : f|_T \widehat{A} \big\}.$$

Here $f|_T \widehat{A}$ denotes that $f|_T g$ for every $g \in \widehat{A}$. By definition, $f|_T \emptyset$ holds for every f, so we have that

$$F(\emptyset) = \mathcal{X} \times I(J(L)) \equiv_w \mathcal{X} \times \big\{ f \in I(J(L)) : f \text{ minimal in } I(J(L)) \big\}.$$

We thus obtain the lattice $F(H)$. Note that H has $\emptyset$ as least element and $I(J(L))$ as largest element. We prove that F is an isomorphism from H to the interval $\big[F(\emptyset), F(I(J(L)))\big] \subseteq \mathfrak{M}_w$. Since H is isomorphic to L this suffices to prove the theorem.

F is injective. Suppose that $A, B \in H$. Note that since A and B are downwards closed, $A = B$ if and only if $\widehat{A} = \widehat{B}$. So it suffices to show that if $\widehat{A} \not\subseteq \widehat{B}$ then $F(A) \not\equiv_w F(B)$. Suppose that $f \in \widehat{A} - \widehat{B}$. Since $f \in I(J(L))$ we have $\{f\} \not\geqslant_w \mathcal{X}$. Since $f \in \widehat{A}$ we also have $\{f\} \not\geqslant_w F(A)$ because $\widehat{A}$ is an antichain. If $f|_T \widehat{B}$ then $\{f\} \geqslant_w F(B)$ so in this case $F(A) \not\equiv_w F(B)$. If $f \not|_T \widehat{B}$ then there is $g \in \widehat{B}$ with

either $f \geqslant_T g$ or $g >_T f$. In the first case we have $f >_T g$ (because g is in $\widehat{B}$ and f is not), hence $\{f\} \geqslant_w F(B)$, and again we can conclude that $F(A) \not\equiv_w F(B)$. In the second case, since $g >_T f \in \widehat{A}$ we have $\{g\} \geqslant_w F(A)$, but $\{g\} \not\geqslant_w F(B)$ because $g \in \widehat{B}$ and $\widehat{B}$ is an antichain, so again $F(A) \not\equiv_w F(B)$.

F is monotone. We claim that $A \subseteq B$ implies that $F(A) \leqslant_w F(B)$. Suppose that $A \subseteq B$ and that $h \in F(B)$. We prove that $\{h\} \geqslant_w F(A)$. We have the following three cases, corresponding to the three components of $F(B)$:

- If $h \in \mathcal{X}$ then we are immediately done.
- If $h \in I(J(L))$, $h|_T\widehat{B}$ then we have one of the following three options:
  - $h|_T\widehat{A}$. In this case we are done immediately.
  - $\exists g \in \widehat{A}\;\; h \geqslant_T g$. In this case we cannot have $h \in \widehat{A}$ because $A \subseteq B$ and $h|_T\widehat{B}$, so we have $h >_T g$ and hence $\{h\} \geqslant_w F(A)$.
  - $\exists g \in \widehat{A}\;\; h \leqslant_T g$. This case cannot occur because A is downwards closed, hence h would be in A, hence in B, contradicting $h|_T\widehat{B}$.
- $h >_T f$ for some $f \in \widehat{B}$. When $f \in \widehat{A}$ we are done. If $f \notin \widehat{A}$ then since $A \subseteq B$ and f is maximal in B we have $f \notin A$, so either $f|_T\widehat{A}$, in which case we are done or $\exists g \in \widehat{A}\; f \geqslant_T g$, in which case $f >_T g$ since $f \notin \widehat{A}$, and again we are done.

$F(A \cap B) \equiv_w F(A) \times F(B)$: By monotonicity of F we have $F(A \cap B) \leqslant_w F(A), F(B)$, hence also $F(A \cap B) \leqslant_w F(A) \times F(B)$. For the other direction $\geqslant_w$, suppose that $h \in F(A \cap B)$. We consider the three cases corresponding to the three components of $F(A \cap B)$.

- If $h \in \mathcal{X}$ then we immediately have that $\{h\} \geqslant_w F(A), F(B)$.
- Suppose that $h \in F(A \cap B)$ because $h >_T f$ for some $f \in \widehat{A \cap B}$. If $f \in \widehat{A}$ or $f \in \widehat{B}$ then $\{f\}' \geqslant_w F(A)$ or $\{f\}' \geqslant_w F(B)$, hence we are done. If $\{h\} \geqslant_w \{g \in I(J(L)) : g|_T\widehat{A}\}$ or $\{h\} \geqslant_w \{g \in I(J(L)) : g|_T\widehat{B}\}$ then we are also done. Otherwise, in particular both $h \not|_T\widehat{A}$ and $h \not|_T\widehat{B}$, say that $g_0 \in \widehat{A}$ and $g_1 \in \widehat{B}$ are such that $h \not|_T g_0$ and $h \not|_T g_1$. It is impossible that $h \leqslant_T g_0, g_1$ because then (because A, B downwards closed) $h \in A \cap B$, contradicting $f \in \widehat{A \cap B}$. So at least one of $g_0 <_T h$ and $g_1 <_T h$ must hold. But in the first case we have $\{h\} \geqslant_w F(A)$ and in the second $\{h\} \geqslant_w F(B)$.
- Finally suppose that $h|_T\widehat{A \cap B}$. When $h|_T\widehat{A}$ or $h|_T\widehat{B}$ then we are done, so suppose that neither of these hold, say $h \not|_T f \in \widehat{A}$ and $h \not|_T g \in \widehat{B}$. When either f or g is in $\widehat{A \cap B}$ then $h \not|_T\widehat{A \cap B}$ contrary to assumption, so we have

that $f,\ g \notin \widehat{A \cap B}$. When $f \geqslant_T g$ then $g \in A \cap B$, and because $g \notin \widehat{A \cap B}$ there is then $h \in \widehat{A \cap B}$ with $h >_T g$, contradicting $g \in \widehat{B}$. Likewise, $g \geqslant_T f$ is impossible, so we have $f|_T g$. Hence either $h >_T f, g$ or $h <_T f, g$. In the latter case $h \in A \cap B$, contradicting $h|_T \widehat{A \cap B}$, and in the former case we have $\{h\} \geqslant_w F(A), F(B)$.

Hence every $h \in F(A \cap B)$ computes an element of either $F(A)$ or $F(B)$.

$F(A \cup B) \equiv_w F(A) + F(B)$: By monotonicity of F we have $F(A), F(B) \leqslant_w F(A \cup B)$, hence also $F(A) + F(B) \leqslant_w F(A \cup B)$. For the other direction, suppose that $\{h\} \geqslant_w F(A), F(B)$. We prove that $\{h\} \geqslant_w F(A \cup B)$. If $\{h\} \geqslant_w \mathcal{X}$ we are immediately done, so assume that $\{h\} \not\geqslant_w \mathcal{X}$. We have to prove that either

$$\exists f \in \widehat{A \cup B} \ \ h >_T f \tag{4}$$

$$\text{or} \quad \exists f|_T \widehat{A \cup B} \ \ h \geqslant_T f. \tag{5}$$

We have the following cases, corresponding to the four remaining ways in which h can be above the components of $F(A)$ and $F(B)$ that are different from $\mathcal{X}$:

- $\{h\} \geqslant_w \{f \in I(J(L)) : f|_T \widehat{A}\}, \{f \in I(J(L)) : f|_T \widehat{B}\}$. Suppose that $h \geqslant_T f_0 \oplus f_1$, $f_0|_T \widehat{A}$ and $f_1|_T \widehat{B}$. Note that it is not possible that $f_0 \oplus f_1 \leqslant_T g$ for some $g \in \widehat{A \cup B}$ because then $f_0 \oplus f_1$ would be below an element of either $\widehat{A}$ or $\widehat{B}$, contradicting that $f_0|_T \widehat{A}$ and $f_1|_T \widehat{B}$. So we either have $f_0 \oplus f_1 >_T g$ for some $g \in \widehat{A \cup B}$ or $f_0 \oplus f_1|_T \widehat{A \cup B}$. In the first case we have $h >_T g$ and we are done by way of (4). In the second case we are done by way of (5).

- $\{h\} \geqslant_w \{f \in I(J(L)) : f|_T \widehat{A}\}$ and $h >_T f_0$ for some $f_0 \in \widehat{B}$. If there is such f_0 with $f_0 \in \widehat{A \cup B}$ then we are done by way of (4), so assume without loss of generality that $f_0 \in \widehat{B} - \widehat{A \cup B}$. If $h|_T \widehat{A \cup B}$ then we are done by (5), so assume that $h \not|_T \widehat{A \cup B}$, say $f \in \widehat{A \cup B}$, $h \not|_T f$. We have one of the following two cases:

  - $h \leqslant_T f$. In this case $f \in \widehat{B}$ is impossible by $f_0 \in \widehat{B}$, so we must have $f \in \widehat{A}$. But this contradicts $\{h\} \geqslant_w \{f \in I(J(L)) : f|_T \widehat{A}\}$.
  - $h >_T f$. In this case we are done by way of (4).

- $\{h\} \geqslant_w \{f \in I(J(L)) : f|_T \widehat{B}\}$ and $h >_T f_0$ for some $f_0 \in \widehat{A}$. This is completely symmetric to the previous case.

- $h >_T f_0, f_1$ for some $f_0 \in \widehat{A}$ and $f_1 \in \widehat{B}$. If either f_0 or f_1 is in $\widehat{A \cup B}$ then we are done by way of (4). Otherwise $f_0, f_1 \notin \widehat{A \cup B}$. When $h|_T \widehat{A \cup B}$ we are done by (5), so assume that $h \not|_T \widehat{A \cup B}$, say $h \not|_T f$ with $f \in \widehat{A \cup B}$. Because $f_0, f_1 \leqslant_T h$ it is impossible that $h \leqslant_T f$, for then either f_0 would not be in $\widehat{A}$ or f_1 would not be in $\widehat{B}$. Hence $h >_T f$, and we are done by way of (4).

F is surjective. It remains to check that if $\mathcal{C} \in \big[F(\emptyset), F(I(J(L)))\big]$ then there is $A \in H$ such that $F(A) = \mathcal{C}$. To this end, let A be a maximal subset of $I(J(L))$ such that $\mathcal{C} \geqslant_w F(A)$. We claim that then also $\mathcal{C} \leqslant_w F(A)$. Namely we have $\mathcal{C} \leqslant_w \mathcal{X}$ because $\mathcal{C} \leqslant_w F(I(J(L))$. As for the other components of $F(A)$, suppose that $f \in I(J(L))$, $f|_T\widehat{A}$ and suppose that $\mathcal{C}$ contains no element of degree $\deg_T(f)$. Then $\mathcal{C} \geqslant_w F(A \cup \{f\})$, for the elements of $\mathcal{C}$ that are mapped to f in the reduction $\mathcal{C} \geqslant_w F(A)$ are all $>_T f$. But $\mathcal{C} \geqslant_w F(A \cup \{f\})$ contradicts the maximality of A. It follows that $\mathcal{C} \leqslant_w \big\{f \in I(J(L)) : f|_T\widehat{A}\big\}$. We also have $\mathcal{C} \leqslant_w \prod\big\{\{f\}' : f \in \widehat{A}\big\}$. Namely suppose not, that is, suppose there is $f \in \widehat{A}$ such that $\mathcal{C} \not\leqslant_w \{f\}'$. Such f cannot be maximal in $I(J(L))$ because $\mathcal{C} \leqslant_w F(I(J(L))$. Hence the set $\{g_0, \ldots, g_m\}$ of all elements of $I(J(L))$ covering f is nonempty. We have $\mathcal{X} \times \{f\}' \equiv_w \mathcal{X} \times \{g_0, \ldots, g_m\}$ because $\mathcal{X}_f \subseteq \mathcal{X}$ and because of the minimality of the g_i over f. But then $\mathcal{C} \geqslant_w F(A \cup \{g_0, \ldots, g_m\})$, contradicting the maximality of A. We have thus proved that $\mathcal{C} \equiv_w F(A)$. This concludes the proof of the surjectivity of F and of the theorem. ☐

Theorem 3.11. *If $J(L)$ is not an initial segment of a finite upper semilattice then L is not isomorphic to an interval in $\mathfrak{M}_w$.*

Proof. We start by arguing as in Theorem 3.7. If $J(L)$ is not an initial segment of a finite upper semilattice then then it must contain two incomparable elements x_0 and x_1 without a least upper bound. If x_0 and x_1 would have no or only one upper bound in $J(L)$ then $J(L)$ could be consistently extended to an upper semilattice, so there must be at least *two* incomparable minimal upper bounds y_0 and y_1 for both x_0 and x_1. But then the poset $\{x_0, x_1, y_0, y_1\}$ is isomorphic to the configuration in Figure 9. (Note that the intervals between the x's and the y's need not be empty though.) Consider the subinterval $[X, Y] \subseteq H(J(L))$, where

$$\begin{aligned} X &= \big\{x \in J(L) : x < x_0 \vee x < x_1\big\}, \\ Y &= \big\{x \in J(L) : x \leqslant y_0 \vee x \leqslant y_1\big\}. \end{aligned}$$

Suppose that $[\mathcal{X}, \mathcal{Y}] \subseteq \mathfrak{M}_w$ is isomorphic to $[X, Y]$ From this assumption we will derive a contradiction.

The lattice $[X, Y]$ starts and ends with a diamond, namely $[X, Y]$ has at the top the diamond with top Y and bottom $Y - \{y_0, y_1\}$ and at the bottom the diamond with top $X \cup \{x_0, x_1\}$ and bottom X. Hence, because by Example 3.2 there is only one way to implement the diamond in $\mathfrak{M}_w$, we can argue as before that there are $f_0, f_1, g_0, g_1 \in \omega^\omega$ with $f_0, f_1 \not\geqslant_w \mathcal{X}$, $f_0|_T f_1$, and $g_0, g_1 \not\geqslant_w \mathcal{Y}$, $g_0|_T g_1$, such that

$$\begin{aligned} \mathcal{X} &\equiv_w \mathcal{Y} \times \{f_0, f_1\}, \\ \mathcal{Y} \times \{f_0\}' \times \{f_1\}' &\leqslant_w \mathcal{Y} \times \{g_0, g_1\}, \\ \mathcal{Y} &\equiv_w \mathcal{Y} \times \{g_0\}' \times \{g_1\}'. \end{aligned}$$

We now have two cases:

Case 1: $g_0, g_1 >_T f_0 \oplus f_1$. Note that the interval

$$\big[X \cup \{x_0, x_1\}, Y - \{y_0, y_1\}\big] \subseteq [X, Y]. \tag{6}$$

under the assumed isomorphism of $[X, Y]$ with $[\mathcal{X}, \mathcal{Y}]$ must correspond to the interval

$$\big[\mathcal{Y} \times \{f_0\}' \times \{f_1\}', \mathcal{Y} \times \{g_0, g_1\}\big] \subseteq [\mathcal{X}, \mathcal{Y}]. \tag{7}$$

Because $f_0 \oplus f_1 \notin \mathcal{Y}$ (since $g_0, g_1 \not\geqslant_w \mathcal{Y}$) we have by Dyments Theorem 1.3 that the subinterval $\big(\mathcal{Y} \times \{f_0 \oplus f_1\}, \mathcal{Y} \times \{f_0 \oplus f_1\}'\big)$ is empty. Hence we see that the element $\mathcal{Y} \times \{f_0 \oplus f_1\}'$ in the interval (7) is join-irreducible in $\mathfrak{M}_w$, and hence also in $[\mathcal{X}, \mathcal{Y}]$. We derive a contradiction by showing that the interval (6) contains only elements that are join-reducible in $[X, Y]$. The elements of this interval are all supersets $A \supseteq X \cup \{x_0, x_1\}$ that because of the special form of $\{x_0, x_1, y_0, y_1\}$ contain no elements both above x_0 and x_1. Hence every such A can always be split into a nonempty part A_0 of elements above x_0 and a nonempty part A_1 of elements above x_1. In particular every element of (6) is join-reducible in $[X, Y]$.

Case 2. If Case 1. does not obtain there must be at least one f incomparable to a g, say that $f_0 |_T g_1$. It is clear that $[X, Y]$ is double diamond-like. We derive a contradiction by showing that $[\mathcal{X}, \mathcal{Y}]$ is not double diamond-like.

Let $\mathcal{H}$ consist of f_0 plus the set of all elements h with $f_1 \leqslant_T h \leqslant_T g_0$ that are minimal with respect to the property $h \not\leqslant_T g_1$. (Note that there can be only finitely many h in $[f_1, g_0]$ since otherwise $[\mathcal{X}, \mathcal{Y}]$ would be infinite, in which case we would have reached a contradiction right away.) Then we have

$$\mathcal{X} \equiv_w \mathcal{Y} \times \{f_0, f_1\} \equiv_w \big(\mathcal{Y} \times \mathcal{H} \times \{g_1\}'\big) \times \big(\mathcal{Y} \times \{f_0\}' \times \{f_1\}\big).$$

We prove that the two elements on the right hand side are join-irreducible in $[\mathcal{X}, \mathcal{Y}]$, and that one is maximal with this property. Since $\mathcal{X}$ is the 0 of the lattice $[\mathcal{X}, \mathcal{Y}]$ this proves that this interval is not double diamond-like.

$\mathcal{Y} \times \{f_0\}' \times \{f_1\}$ *is join-irreducible:* When $\mathcal{A} \in [\mathcal{X}, \mathcal{Y}]$ and $\mathcal{A} <_w \mathcal{Y} \times \{f_0\}' \times \{f_1\}$ then $\mathcal{A} \leqslant_w \mathcal{Y} \times \{f_0\} \times \{f_1\}$: We have $\mathcal{X} \equiv_w \mathcal{Y} \times \{f_0\} \times \{f_1\} \leqslant_w \mathcal{A}$, and if in this reduction all elements of $\mathcal{A}$ that are mapped to f_0 are strictly above it then $\mathcal{Y} \times \{f_0\}' \times \{f_1\} \leqslant_w \mathcal{A}$, contradicting the assumption. Hence $\mathcal{A}$ contains an element of $\deg_T(f_0)$, and because $\mathcal{A} \leqslant_w \mathcal{Y} \times \{f_1\}$ we then have $\mathcal{A} \leqslant_w \mathcal{Y} \times \{f_0\} \times \{f_1\}$.

$\mathcal{Y} \times \mathcal{H} \times \{g_1\}'$ *is a maximal join-irreducible element of* $[\mathcal{X}, \mathcal{Y}]$*:* Join-reducibility is seen with an argument similar to the previous one: When $\mathcal{A} \in [\mathcal{X}, \mathcal{Y}]$ and $\mathcal{A} <_w \mathcal{Y} \times \mathcal{H} \times \{g_1\}'$ then $\mathcal{A} \leqslant_w \mathcal{Y} \times \mathcal{H} \times \{g_1\}$. Namely, we clearly have $\mathcal{A} \leqslant_w \mathcal{Y} \times \mathcal{H}$. If $\mathcal{A}$ would not have an element of degree $\deg_T(g_1)$ then contrary to assumption we would have $\mathcal{A} \geqslant_w \mathcal{Y} \times \mathcal{H} \times \{g_1\}'$: By definition of $\mathcal{H}$ and because $\mathcal{A} \geqslant_w \mathcal{X} \equiv_w \mathcal{Y} \times \{f_0, f_1\}$, any element in $\mathcal{A}$ that is not below g_1 can be mapped to either $\mathcal{Y}$ or $\mathcal{H}$. Any element in $\mathcal{A}$ that can be mapped to g_1 is actually strictly above g_1, so can be mapped to $\{g_1\}'$ by the identity. Hence we also have $\mathcal{A} \leqslant_w \{g_1\}$.

To see the maximality: Suppose $\mathcal{A} \in [\mathcal{X}, \mathcal{Y}]$ is such that $\mathcal{A} >_w \mathcal{Y} \times \mathcal{H} \times \{g_1\}'$. Then $\mathcal{A}$ is of the form $\mathcal{A} = \mathcal{Y} \times \mathcal{K} \times \{g_1\}'$, with $\{g_0\}' \geqslant_w \mathcal{K} >_w \mathcal{H}$. But then it is easy to check that

$$\mathcal{A} \equiv_w \big(\mathcal{Y} \times \mathcal{H} \times \{g_1\}'\big) + \big(\mathcal{Y} \times \mathcal{K} \times \{g_1\}\big),$$

using that $\mathcal{H} \oplus \{g_1\} \subseteq \{g_1\}'$. The two components on the right hand side are w-incomparable and in $[\mathcal{X}, \mathcal{Y}]$, so $\mathcal{A}$ is join-reducible in $[\mathcal{X}, \mathcal{Y}]$. This concludes Case 2.

Summarizing, we have seen that $H(J(L))$ contains a subinterval $[X, Y]$ that cannot be an interval in $\mathfrak{M}_w$. Thus $H(J(L))$, and hence by by Theorem 3.5 also L, cannot be an interval of $\mathfrak{M}_w$. □

By combining the above results we obtain the following characterization of the finite intervals of $\mathfrak{M}_w$:

Theorem 3.12. *For any finite distributive lattice L the following are equivalent:*

1. *L is isomorphic to an interval in $\mathfrak{M}_w$,*
2. *$J(L)$ is an initial segment of a finite upper semilattice,*
3. *L has no double diamond-like lattice as a subinterval.*

Proof. Item 2. and 3. are equivalent by Theorem 3.7. They imply item 1. by Theorem 3.10. Conversely, 1. implies 2. by Theorem 3.11. □

Corollary 3.13. *A finite distributive lattice is an initial segment of $\mathfrak{M}_w$ if and only if it has no double diamond-like subinterval and it has a meet-irreducible 0.*

Proof. We can extend the definition of the mapping F in the proof of Theorem 3.12 as follows. Define $\mathcal{X}_f$ as before and let

$$\mathcal{X}_0 = \big\{h \in \omega^\omega : h|_T f \text{ for all } f \text{ minimal in } I(J(L))\big\},$$

$$\mathcal{X} = \mathcal{X}_0 \cup \bigcup_{f \in I(J(L))} \mathcal{X}_f.$$

Then for every $A \in H$ define $F(A)$ as before, using this new definition of $\mathcal{X}$. This addition does not change anything in the proof of Theorem 3.10, but now we have that $F(\emptyset) \equiv_w 0'$, as is easily checked, using that we chose the minimal elements of $I(J(L))$ of minimal T-degree. Thus we obtain that a finite distributive lattice has no double diamond-like subinterval if and only if it is isomorphic to an interval of the form $[\mathbf{0}', \mathbf{A}]$ in $\mathfrak{M}_w$. From this the corollary follows immediately. □

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